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On the numerical solution to two fluid models via a cell centered finite volume method

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Abstract – A new method for the discretization of nonlinear systems of partial differential equations occurring in the numerical simulation of two phase flows is proposed. This method is based on a cell centered finite volume discretization on possibly unstructured meshes and aims to approximate three-dimensional stationary and evolution problems in arbitrary geometries. We are able to consider conservative and non-conservative systems of equations and the method belongs to the class of shock-capturing upwind ones. In the paper we put the emphasis on the treatment of terms involving first-order derivatives since we deal with the change of type (hyperbolic to non-hyperbolic). One of the features of the method is that it does not rely a priori on the hyperbolic character of the convection operator. The method is illustrated on a classical numerical benchmark and we refer to the bibliography concerning various and numerous applications in the context of two phase flows. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

One phase fluid mechanics remains an open problem due to the classical problem of turbulence. However, there is agreement on the fact that Navier–Stokes equations govern local and instantaneous variables such as velocity, pressure and temperature.

In the two phase flow problem, such a consensus does not exist yet. The derivation of the 'right' model is far from being achieved, besides the fact that rigorous and systematic studies have been undertaken (Boure and Delhaye [1]; Drew and Lahey [2]; Drew and Wallis [3]; Ishii [4]; Ransom [5]), the degree of complexity of these models makes the solution practically not reachable by numerical computations. The essential difficulty is to describe the turbulent interfacial geometry between the two phases and take into account steep gradients of the variables across the interfaces in order to determine the mass, momentum and energy transfers.

This leads one to consider what is known as averaged equations. This averaging is done in space and time, and leads to the appearance of a new variable, the 'void fraction', which characterizes the volume occupied by one of the two phases per unit volume of the mixture. The transfers are considered like those between the non-equilibrium states and the equilibrium state.

Many computer tools have been developed to solve the two phase flow problem in a great number of industrial applications. The models which are used in these calculations are based on total conservation of mass, momentum and energy and some of them take into account thermal or/and mechanical non-equilibrium transfers by simple correlation. However, the reliability of the numerical simulations is found on the qualification of models from numerous experimental tests. This looks like a 'black box' approach where

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little information of knowledge is contained in the model but many correlations are built on experimental tests. And if the predictions given by this model are correct into the domain of application defined by the experimental tests conditions, extrapolations are not recommended.

The finding of a unique model for multiphase flow remains a problem of fundamental research. However practical three-dimensional models exist in order to simulate the evolution of parameters which characterize a multiphase flow. These models can be classified into various groups depending on the deviation of temperature and/or velocity of each phase with regards to the equilibrium state. This equilibrium state is defined as a state where the temperature and the velocity of each phase are equal (for given specific volume, momentum and total energy).

At least if a model is supposed to be suitable to simulate a two phase flow, the same numerical method used in one phase flow should be fitted to obtain the solution. Up to now, no numerical method is convenient because some do not respect the balance, others damp down strong gradients and seldom do these numerical methods follow good consistency and have convergence properties.

The numerical techniques can be classified into 3 categories: finite differences, finite elements and finite volumes. If all of these techniques have been used to calculate the solution of two phase flow, the finite volumes method seems to be the most appropriate. Indeed this technique has good properties of conservation of mass, momentum and energy balance and altogether this approach looks like the physicist's approach to observe the natural phenomena.

Some serious difficulties arise in using finite volumes and up to now only the efficient staggered mesh method has been used. However, the staggered meshes techniques are tedious to code and lead to somewhat cumbersome data structures on non-structured meshes.

To summarize, the ideal numerical method should be simple in order to apply to any model, from equilibrium model to non-equilibrium model, conservative in order to respect the fundamental conservation physical laws, flexible in order to be applied on any mesh type such as structured, non-structured and non-conform types and of course at least cost of computational time.

In the past ten years, there have been a large number of 'modern' numerical methods that have been developed for two phase flow computations. Roughly speaking, one of the main goals was to try to transfer the shock-capturing upwind schemes (Roe [6]; van Leer [7]; Osher and Solomon [8]) from one phase CFD to two phase CFD. Most of these methods were based on Godunov type methods such as exact or approximate Riemann solvers ([9–19]) or flux splitting techniques ([20,21]). Since these works originated from methods devised in the context of the Euler equation of gas dynamics, the hyperbolicity of the convection operator was considered as a necessary condition. But, as it is well known (and we shall discuss that in details in this paper), the models occurring in two phase CFD are not necessarily hyperbolic. As a consequence of that most of the methods derived in the previous articles require hyperbolic equations. So, in the case where the models were not hyperbolic, corrections terms were added in order to make them so (at least in the numerical range where the computations were performed).

We would like to point out two facts.

(i) Firstly, if we want to be able to determine whether hyperbolicity is necessary in order to achieve stable and convergent numerical computations, we need to use a method that allows us to consider both hyperbolic and non-hyperbolic convection operators.

(ii) Secondly, in actual computations, the physical model contains also viscous and diffusive terms (Navier–Stokes like models) and even turbulent diffusion contributions. It has been shown that molecular diffusion and thermal diffusivity are indeed sufficient in order to make the evolution problem well posed, either in the linearized sense (Arai [22], Ramos [23]) or in the nonlinear sense for small data

(Ramos [23]). In fact including these terms leads to a mixed hyperbolic–parabolic equation (recall that there is no diffusion on the mass balance equation).

Our goal in this paper is to propose a shock-capturing method that does not a priori rely on hyperbolicity properties of the convection operator. We also impose that the method should be applicable on unstructured finite volume meshes in 3 spatial dimensions and not limited by CFL constraints (implicit schemes). This method was first presented in a short note in 1995, [24]. Since it has been validated and extended, in the framework of a collaboration between the Département Transferts Thermiques et Aérodynamique (Electricité de France) and the Centre de Mathématiques et de Leurs Applications, on numerous test cases. We refer to the report [25] concerning both numerical results and references including numerical results.

In the present paper, we are going to focus on the application of our method to two fluid models, but it has a broader applicability. For instance the method applies naturally to the classical Euler equations for gas dynamics (in this case the scheme is very close to Roe's scheme, see Remark 3). The method also applies to the so called Homogeneous Equilibrium Model which is also of use in the context of two phase flow. In de Vuyst et al. [26], it is shown that the method is able to capture a condensation shock in which a phase transition occurs from a diphasic mixture to a single phase (liquid). In this transition, the fluid is compressible upstream and almost incompressible downstream and the ratio of the density before and after the shock is of the order of 15. Although this problem was known for a while, its numerical solution was not found up to now in the setting of the simple and natural framework of the Homogeneous Equilibrium Model.

Let us consider a two fluid model which reads as follows (see, e.g., Stewart and Wendroff [27] and Ishii [4]):

$$\frac{\partial(\alpha_1\rho_1)}{\partial t} + \operatorname{div}(\alpha_1\rho_1 u_1) = 0, \tag{1}$$

$$\frac{\partial(\alpha_2\rho_2)}{\partial t} + \operatorname{div}(\alpha_2\rho_2 u_2) = 0, \tag{2}$$

$$\frac{\partial(\alpha_1\rho_1 u_1)}{\partial t} + \operatorname{div}\big(\alpha_1(\rho_1 u_1 \otimes u_1 + p\mathbf{I})\big) - p\nabla\alpha_1 = \alpha_1\rho_1 g, \tag{3}$$

$$\frac{\partial(\alpha_2\rho_2 u_2)}{\partial t} + \operatorname{div}\big(\alpha_2(\rho_2 u_2 \otimes u_2 + p\mathbf{I})\big) - p\nabla\alpha_2 = \alpha_2\rho_2 g, \tag{4}$$

$$\frac{\partial(\alpha_1\rho_1 E_1)}{\partial t} + \operatorname{div}(\alpha_1\rho_1 H_1 u_1) + p\frac{\partial\alpha_1}{\partial t} = \alpha_1\rho_1 g \cdot u_1, \tag{5}$$

$$\frac{\partial(\alpha_2\rho_2 E_2)}{\partial t} + \operatorname{div}(\alpha_2\rho_2 H_2 u_2) + p\frac{\partial\alpha_2}{\partial t} = \alpha_2\rho_2 g \cdot u_2. \tag{6}$$

This model is simplified in the sense that we have omitted in its right-hand side

- most of the terms which are related with transfers between the two fluids 1 and 2,
- the terms which are related with dissipative phenomena,

and we refer to section 5 concerning these purposes. The terms that have been omitted might be important in most of two-phase flows. These terms involve lower-order and higher-order terms who are inconsequential on the type of the equation (hyperbolic or non-hyerbolic). Equations (1) to (6) will be named as the basic model.

Let us now describe the physical meaning of each variable: α_i is the volume fraction of the fluid i, ρ_i is the density of the fluid i, u_i denotes the velocity of the phase i and p is the thermodynamic pressure. Denoting by

e_i the specific internal energy of the phase i, we have set $E_i = e_i + \frac{1}{2}|u|^2$: the total specific energy of the fluid i and $H_i = E_i + p/\rho_i$ the total specific enthalpy of the fluid i (we shall also use the notation $h_i \equiv e_i + p/\rho_i$ for the specific enthalpy of the fluid i). Gravity is denoted by g.

We have the relation $\alpha_1 + \alpha_2 = 1$ and in order to close the system (1) to (6), we have to write two equations of state:

$$F_i(p, \rho_i, e_i) = 0, \quad i = 1, 2. \tag{7}$$

An isentropic version of this system of equation can be obtained as follows. Introducing the specific entropy of the fluid i, s_i, defined by:

$$T_i \, \mathrm{d}s_i = \mathrm{d}e_i - \frac{p}{\rho_i^2} \, \mathrm{d}\rho_i, \tag{8}$$

and assuming that there is no production of entropy into the shocks, equations (5) and (6) lead to:

$$\frac{\partial(\alpha_1\rho_1 s_1)}{\partial t} + \mathrm{div}(\alpha_1\rho_1 s_1 u_1) = 0, \tag{9}$$

$$\frac{\partial(\alpha_2\rho_2 s_2)}{\partial t} + \mathrm{div}(\alpha_2\rho_2 s_2 u_2) = 0. \tag{10}$$

In view of (1) and (2), we can have solutions with constant entropies: i.e. s_1 and s_2 are constant. The reduced system which is then obtained is (1), (2), (3) and (4) and the equations of states (7) are replaced by isentropic ones:

$$G_i(p, \rho_i) = 0, \quad i = 1, 2. \tag{11}$$

Remark 1: (i) The model (1)–(6) is a 'one pressure' model. That is, the two fluids share the same pressure. There are also models with two pressures p_1 and p_2 (see Stewart and Wendroff [27], Ransom and Hicks [28]). In such a case, an extra equation (differential or algebraic) must be added. In most cases this new equation is related to a relaxation process which tends to equalize the two pressures.

(ii) Usually, equations (3) and (4) are written as follows:

$$\frac{\partial(\alpha_1\rho_1 u_1)}{\partial t} + \mathrm{div}(\alpha_1\rho_1 u_1 \otimes u_1) + \alpha_1 \nabla p = \alpha_1\rho_1 g, \tag{12}$$

$$\frac{\partial(\alpha_2\rho_2 u_2)}{\partial t} + \mathrm{div}(\alpha_2\rho_2 u_2 \otimes u_2) + \alpha_2 \nabla p = \alpha_2\rho_2 g. \tag{13}$$

These two forms are not equivalent when one considers non smooth solutions. In this last form, the physical flux of momentum does not involve the pressure and we believe that the first one is more relevant from the physical point of view, see the discussion in section 3.1.1. Actually, see e.g. Ishii [4], (3) and (4) correspond to local instantaneous formulation (the nonconservative terms appear after averaging and are due to transfer between phases). Moreover this form has the property that the fluxes which are involved produce the single phase fluxes in the case where α_i equals 0 or 1.

1.1. Hyperbolicity of the 6 equation model

Let us consider the one-dimensional case. The model (1)–(6) can be written under a quasilinear form:

$$v_t + A(v)v_x = S(v), \tag{14}$$

where v and $S(v)$ are the following vectors:

$$v = (\alpha_1\rho_1, \alpha_2\rho_2, \alpha_1\rho_1 u_1, \alpha_2\rho_2 u_2, \alpha_1\rho_1 E_1, \alpha_2\rho_2 E_2), \tag{15}$$

$$S(v) = (0, 0, \alpha_1\rho_1 g, \alpha_2\rho_2 g, \alpha_1\rho_1 g \cdot u_1, \alpha_2\rho_2 g \cdot u_2), \tag{16}$$

and $A(v)$ is a 6×6 matrix. As it is classical, we shall name $A(v)v_x$ as the convection operator. The system (14) (or the convection operator) is said to be hyperbolic (by definition) if the matrix $A(v)$ can be diagonalized on $\mathbb{R}$. In general it is not the case, see sections 3.2.2 and 3.2.4.

1.2. Content

The paper is organized as follows. The next section is devoted to a detailed presentation of our method. In section 3 we show how this numerical scheme applies to the basic two fluid model. Then, section 4 is devoted to a numerical illustration. Finally, section 5 indicates some further developments. We have also included an Appendix dealing with some technical matters.

2. Discretization in the finite volume framework

In this section we present first our method for the discretization of systems like (1)–(6) in the one-dimensional case. The extension to the multidimensional case is discussed in section 2.3. We begin with the explicit scheme and then give the implicit scheme in section 2.2.

2.1. The one-dimensional case

Let us consider general systems that can be written as follows:

$$v_t + f(v)_x + \tilde{C}(v)v_x + D(v)v_t = \tilde{S}(v), \tag{17}$$

here $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$, f maps $\mathbb{R}^m$ into itself, $C(v)$ and $D(v)$ are $m \times m$ matrices and S maps $\mathbb{R}^m$ into itself.

Example 1: The system (1) to (6) enters in this category where v (resp. $\tilde{S}(v)$) is defined in (15) (resp. (16)) and $f(v)$, $\tilde{C}(v)v_x$ and $D(v)v_t$ are as follows:

$$f(v) = \big(\alpha_1\rho_1 u_1, \alpha_2\rho_2 u_2, \alpha_1\big(\rho_1 u_1^2 + p\big), \alpha_2\big(\rho_2 u_2^2 + p\big), \alpha_1\rho_1 H_1 u_1, \alpha_2\rho_2 H_2 u_2\big), \tag{18}$$

$$\tilde{C}(v)v_x = \big(0, 0, -p(\alpha_1)_x, -p(\alpha_2)_x, 0, 0\big), \tag{19}$$

$$D(v)v_t = \big(0, 0, 0, 0, p(\alpha_1)_t, p(\alpha_2)_t\big). \tag{20}$$

2.1.1. The conservative case

Let us first consider equation (17) when $\tilde{C}(v) \equiv 0$, $D(v) \equiv 0$ and $\tilde{S}(v) \equiv 0$ (e.g. the Euler equation for perfect fluids enters in this category). That is we address

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = 0, \tag{21}$$

where $v \in \mathbb{R}^m$ and $f : \mathbb{R}^m \mapsto \mathbb{R}^m$. We denote by $A(v)$ the Jacobian matrix $\partial f(v)/\partial v$ and we deal first with the case where (21) is smoothly hyperbolic that is to say: for every v there exists a smooth basis $(r_1(v), \ldots, r_m(v))$ of $\mathbb{R}^m$ consisting of eigenvectors of $A(v)$: $\exists \lambda_k(v) \in \mathbb{R}$ such that $A(v)r_k(v) = \lambda_k(v)r_k(v)$. It is then possible to construct $(l_1(v), \ldots, l_m(v))$ such that ${}^t A(v) l_k(v) = \lambda_k(v) l_k(v)$ and $l_k(v) \cdot r_p(v) = \delta_{k,p}$.

Let $\mathbb{R} = \bigcup_{j\in\mathbb{Z}} [x_{j-1/2}, x_{j+1/2}]$ be a one-dimensional mesh. Our goal is to discretize (21) by a finite volume method. We set $\Delta x_j \equiv x_{j+1/2} - x_{j-1/2}$, $\Delta t_n \equiv t_{n+1} - t_n$ (we also have $\mathbb{R}_+ = \bigcup_{n\in\mathbb{N}} [t_n, t_{n+1}]$) and

$$\tilde{v}_j^n \equiv \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} v(x, t_n)\, \mathrm{d}x, \qquad \tilde{f}_{j+1/2}^n \equiv \frac{1}{\Delta t_n} \int_{t_{n+1}}^{t_n} f\big(v(x_{j+1/2}, t)\big)\, \mathrm{d}t.$$

With these notations, we deduce from (21) the *exact* relation:

$$\tilde{v}_j^{n+1} = \tilde{v}_j^n - \frac{\Delta t_n}{\Delta x_j} \big(\tilde{f}_{j+1/2}^n - \tilde{f}_{j-1/2}^n\big). \tag{22}$$

Since the $(\tilde{f}_{j+1/2}^n)_{j\in\mathbb{Z}}$ cannot be expressed in terms of the $(\tilde{v}_j^n)_{j\in\mathbb{Z}}$ one has to make an approximation. In order to keep a compact stencil, it is more efficient to use a three points scheme: the physical flux $\tilde{f}_{j+1/2}^n$ is approximated by a numerical flux $g_j^n(v_j^n, v_{j+1}^n)$. Let us show how we construct this flux here. We observe that since $A(v)\frac{\partial v}{\partial t} = \frac{\partial f(v)}{\partial t}$ then according to (21),

$$\frac{\partial f(v)}{\partial t} + A(v)\frac{\partial f(v)}{\partial x} = 0. \tag{23}$$

This shows that the flux $f(v)$ is advected by $A(v)$ (like v since we also have $\frac{\partial v}{\partial t} + A(v)\frac{\partial v}{\partial x} = 0$). The numerical flux $g_j^n(v_j^n, v_{j+1}^n)$ represents the flux at an interface. Using a mean value $\mu_{j+1/2}^n$ of v at this interface, we replace (23) by the linearization:

$$\frac{\partial f(v)}{\partial t} + A\big(\mu_{j+1/2}^n\big)\frac{\partial f(v)}{\partial x} = 0. \tag{24}$$

It follows that, defining the k-th characteristic flux component to be $f_k(v) \equiv l_k(\mu_{j+1/2}^n) \cdot f(v)$, we have

$$\frac{\partial f_k(v)}{\partial t} + \lambda_k\big(\mu_{j+1/2}^n\big)\frac{\partial f_k(v)}{\partial x} = 0. \tag{25}$$

This linear equation can be solved explicitly now and we have:

$$f_k(v)(x,t) = f_k(v)\big(x - \lambda_k\big(\mu_{j+1/2}^n\big)(t - t_n), t_n\big). \tag{26}$$

From this equation it is then natural to introduce the following definition.

DEFINITION 1: *For the conservative system* (21)*, at the interface between the two cells* $[x_{j-1/2}, x_{j+1/2}]$ *and* $[x_{j+1/2}, x_{j+3/2}]$*, the characteristic flux* g^{CF} *is defined by the following formula* (*we take* $\mu_{j+1/2}^n \equiv (\Delta x_j v_j^n + \Delta x_{j+1} v_{j+1}^n)/(\Delta x_j + \Delta x_{j+1})$)*: for* $k \in \{1, \ldots, m\}$*,*

$$\begin{aligned}
&l_k\big(\mu_{j+1/2}^n\big) \cdot g_j^{CF,n}\big(v_j^n, v_{j+1}^n\big) = l_k\big(\mu_{j+1/2}^n\big) \cdot f\big(v_j^n\big), \quad \text{when } \lambda_k\big(\mu_{j+1/2}^n\big) > 0,\\
&l_k\big(\mu_{j+1/2}^n\big) \cdot g_j^{CF,n}\big(v_j^n, v_{j+1}^n\big) = l_k\big(\mu_{j+1/2}^n\big) \cdot f\big(v_{j+1}^n\big), \quad \text{when } \lambda_k\big(\mu_{j+1/2}^n\big) < 0,\\
&l_k\big(\mu_{j+1/2}^n\big) \cdot g_j^{CF,n}\big(v_j^n, v_{j+1}^n\big) = l_k\big(\mu_{j+1/2}^n\big) \cdot \left(\frac{f(v_{j+1}^n) + f(v_j^n)}{2}\right),
\end{aligned} \tag{27}$$

when $\lambda_k(\mu^n_{j+1/2}) = 0$.

Remark 2: At first glance, the derivation of (23) from (21), is only valid for continuous solutions since $A(v)\frac{\partial f(v)}{\partial x}$ is a non-conservative product. In fact equation (23) can be justified even in the case of shocks as is proved in Ghidaglia [29]. Let us briefly recall here the key point. Assuming that the solution undergoes a discontinuity along a family of disjoint curves, we can focus on one of these curves that we parameterize by the time variable t. Hence, locally, on each side of this curve, $v(x,t)$ is smooth and jumps across the curve $x = \Sigma(t)$. The Rankine–Hugoniot condition implies that $f(v(x,t)) - \sigma(t)v(x,t)$, where $\sigma(t) \equiv \mathrm{d}\Sigma(t)/\mathrm{d}t$, is smooth across the discontinuity curve and therefore $A(v)\frac{\partial f(v)}{\partial x}$ can be defined as $A(v)\frac{\partial f(v)}{\partial x} \equiv A(v)\frac{\partial (f(v)-\sigma v)}{\partial x} + \sigma\frac{\partial f(v)}{\partial x}$.

PROPOSITION 1: *Formula* (27) *can be written as follows*: $g^{CF,n}_j(v^n_j, v^n_{j+1}) = g^{CF}(\mu^n_j; v^n_j, v^n_{j+1})$ *where*

$$g^{CF}(\mu; v, w) \equiv \sum_{\lambda_k(\mu)<0} \big(l_k(\mu)\cdot f(w)\big) r_k(\mu) + \sum_{\lambda_k(\mu)=0} \left(l_k(\mu)\cdot \frac{f(v)+f(w)}{2}\right) r_k(\mu) + \sum_{\lambda_k(\mu)>0} \big(l_k(\mu)\cdot f(v)\big) r_k(\mu). \tag{28}$$

Proof. – This comes from the useful identity valid for all vectors Φ and μ in $\mathbb{R}^m$: $\Phi = \sum_{k=1}^{k=m} (l_k(\mu)\cdot\Phi) r_k(\mu)$. We also observe that (28) can be written under the following condensed form:

$$g^{CF}(\mu; v, w) = \frac{f(v)+f(w)}{2} - U(\mu; v, w)\frac{f(w)-f(v)}{2}, \tag{29}$$

where $U(\mu; v, w)$ is the sign of the matrix $A(\mu)$ which is defined by

$$\operatorname{sgn}(A(\mu))\Phi = \sum_{k=1}^{k=m} \operatorname{sgn}(\lambda_k)(l_k(\mu)\cdot\Phi) r_k(\mu).$$

The form (29) refers to what we have called a numerical flux leading to a flux scheme ([29]). □

Remark 3: Let us discuss the relation, *in the conservative case*, between the characteristic numerical flux g^{CF} and the numerical flux leading to Roe's scheme ([6]). This later scheme relies on an algebraic property of the continuous flux $f(v)$ which is as follows. It is assumed that for all admissible states v and w, there exists a $m \times m$ matrix $A^{\mathrm{ROE}}(v, w)$ such that $f(v) - f(w) = A^{\mathrm{ROE}}(v, w)(v - w)$ (Roe's identity). Then the numerical flux leading to Roe's scheme is given by:

$$g^{\mathrm{ROE}}(v, w) = \frac{f(v)+f(w)}{2} - \big|A^{\mathrm{ROE}}(v, w)\big|\frac{w-v}{2}. \tag{30}$$

However, using Roe's identity, we obtain that

$$g^{\mathrm{ROE}}(v, w) = \frac{f(v)+f(w)}{2} - \operatorname{sgn}\big(A^{\mathrm{ROE}}(v, w)\big)\frac{f(w)-f(v)}{2}, \tag{31}$$

which is of the form (29): Roe's scheme is also a flux scheme. The characteristic flux proposed in this paper is more versatile than Roe's scheme in the sense that it does not rely on an algebraic property of the flux. Hence for complex systems (like those encountered in the context of two phase flows) this scheme appears like an efficient generalization of Roe's scheme. Moreover, as we shall see below, this scheme has a natural generalization to

arbitrary non-conservative systems. Finally, the fact that the numerical flux is a linear combination of the two fluxes induces a quite weak dependance on the state μ which appears in formula (28), see Cortes and Ghidaglia [30].

Combining (22) and (27), we arrive to the explicit scheme:

$$v_j^{n+1} = v_j^n - \frac{\Delta t_n}{\Delta x_j}\big(g_j^{CF,n}(v_j^n, v_{j+1}^n) - g_j^{CF,n}(v_{j-1}^n, v_j^n)\big). \tag{32}$$

Let us now discuss the case where $A(v)$ has complex eigenvalues. In such a case, they occur by pairs: $\lambda_k(v) + i\,\chi_k(v)$ and $\lambda_{k+1}(v) + i\,\chi_{k+1}(v) = \lambda_k(v) - i\,\chi_k(v)$. Then (25) is replaced by the system:

$$\frac{\partial f_k(v)}{\partial t} + \lambda_k(\mu_{j+1/2}^n)\frac{\partial f_k(v)}{\partial x} + \chi_k(\mu_{j+1/2}^n)\frac{\partial f_{k+1}(v)}{\partial x} = 0, \tag{33}$$

$$\frac{\partial f_{k+1}(v)}{\partial t} + \lambda_k(\mu_{j+1/2}^n)\frac{\partial f_{k+1}(v)}{\partial x} - \chi_k(\mu_{j+1/2}^n)\frac{\partial f_k(v)}{\partial x} = 0. \tag{34}$$

Here again $\lambda_k(\mu_{j+1/2}^n)$ plays the role of an advection velocity. For that reason we keep formulas (27) to (29) unchanged, and this produces real numerical fluxes. Finally, we discuss the case where the matrix $A(v)$ is not diagonalizable, i.e. when there is at least one Jordan block in its reduced form. In such a case, each block is associated with a single eigenvalue and we extend the definitions above by making use of the sign of the real part of this eigenvalue.

2.1.2. The non-conservative case

Let us now consider the full equation (17). First we invert the matrix $\mathbf{I} + D(v)$. If this matrix is not invertible, then (17) is not an evolution equation and therefore the system must be transformed and the evolutionary variable v must be changed (this occurs for example in incompressible fluid flows where the pressure is not an evolutionary variable). Now inverting $\mathbf{I} + D(v)$ and setting:

$$C(v) \equiv \big(\mathbf{I} + D(v)\big)^{-1}\big(\tilde{C}(v) + J(v)\big) - J(v), \tag{35}$$

where $J(v)$ denotes the Jacobian matrix $\partial f(v)/\partial v$, and

$$S(v) \equiv \big(\mathbf{I} + D(v)\big)^{-1}\tilde{S}(v), \tag{36}$$

we arrive at

$$v_t + f(v)_x + C(v)v_x = S(v). \tag{37}$$

Remark 4: One of the features of non-conservative models like (37) is the intrinsic non-uniqueness in the writing of the convective terms. Indeed, we can modify the term $f(v)_x + C(v)v_x$ by adding to $f(v)$ any function $g(v,t)$ provided we subtract from the matrix $C(v)$ the matrix $\partial g(v,t)/\partial v$:

$$\big(f(v) + g(v,t)\big)_x + \left(C(v) - \frac{\partial g(v,t)}{\partial v}\right)v_x = f(v)_x + C(v)v_x. \tag{38}$$

At the continuous level and for smooth solutions, this makes no difference, in contrast with the discrete case where each form will lead to a different solution. In the case where the underlying exact solution is not smooth (presence of shocks), by changing the writing of the convective terms, one can obtain at convergence different

solutions. It is a well known fact for non-conservative equations. From our point of view this simply means that the splitting of the convective terms (between conservative and non-conservative) must rely on supplementary information (like, e.g., physical ones).

Let us return to the discretization of (37). We have to discretize 3 terms: (i) $f(v)_x$, (ii) $C(v)v_x$ and (iii) $S(v)$. Concerning the last term we simply take (we refer to [31] concerning a full discussion on the discretization of this term):

$$\frac{1}{\Delta x_j}\int_{x_{j-1/2}}^{x_{j+1/2}} S(v) \approx S\big(v_j^n\big). \tag{39}$$

Concerning the first term, we observe that the analogue of (23) is now:

$$\frac{\partial f(v)}{\partial t} + \tilde{A}(v)\frac{\partial f(v)}{\partial x} = J(v)S(v), \tag{40}$$

where:

$$\tilde{A}(v) \equiv J(v) + C(v)J(v)^{-1} \quad \text{(we recall that } J(v) \equiv \frac{\partial f(v)}{\partial v}\text{)}. \tag{41}$$

Remark 5: In the case where $C(v) = 0$, the conservative case, we have $A(v) = J(v)$ and, as already observed, v is advected by $J(v)$. Here, v is advected by $A(v) \equiv J(v) + C(v)$ but $f(v)$ is advected by $\tilde{A}(v)$. These two advection matrices are conjugate since $\tilde{A}(v) = J(v)A(v)J(v)^{-1}$.

The discussion that followed (23) leads us then to replace (29) with the following definition.

DEFINITION 2: *For the non-conservative system* (37)*, the characteristic flux* g^{CF} *is defined by the following formula*:

$$g^{CF}(\mu; v, w) = \frac{f(v) + f(w)}{2} - U(\mu; v, w)\frac{f(w) - f(v)}{2}, \tag{42}$$

where $U(\mu; v, w)$ *is the sign of the matrix* $\tilde{A}(\mu)$.

It remains to discuss the discretization of the second term, namely $C(v)v_x$. We write $C(v)v_x = E(v)f(v)_x$ with $E(v) \equiv C(v)J(v)^{-1}$ and make the following approximation:

$$\begin{aligned}\frac{1}{\Delta x_j \Delta t_n}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t_{n+1}}^{t_n} C(v)v_x \,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{\Delta x_j \Delta t_n}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t_{n+1}}^{t_n} E(v)f(v)_x \,\mathrm{d}x\,\mathrm{d}t \\ &\approx E\big(v_j^n\big)\frac{1}{\Delta x_j \Delta t_n}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t_{n+1}}^{t_n} f(v)_x \,\mathrm{d}x\,\mathrm{d}t \\ &= E\big(v_j^n\big)\frac{1}{\Delta t_n}\int_{t_{n+1}}^{t_n}\big(f\big(v(x_{j+1/2}, t)\big) - f\big(v(x_{j-1/2}, t)\big)\big)\,\mathrm{d}t. \end{aligned} \tag{43}$$

Hence combining (39), (42) and (43), we arrive at the following explicit scheme for the discretization of (37):

$$v_j^{n+1} = v_j^n - \frac{\Delta t_n}{\Delta x_j}\big(\mathbf{I} + E\big(v_j^n\big)\big)\big(g_j^{CF,n}\big(v_j^n, v_{j+1}^n\big) - g_j^{CF,n}\big(v_{j-1}^n, v_j^n\big)\big) + S\big(v_j^n\big). \tag{44}$$

Remark 6: In Remark 3, we discussed the relation, in the conservative case, between our characteristic flux and Roe's numerical flux. Toumi and Kumbaro [10], and Toumi [12] have generalized Roe's scheme to the context of non-conservative systems occurring in two phase flows. Again Roe's scheme requires some algebraic

properties, while ours does not. We refer to the paper of Ghidaglia, le Coq and Toumi [32] concerning the comparison of these two schemes.

2.2. The time-implicit discretization

The previous scheme is based on the solution to the characteristic equation (25) given by (26). The value taken for the flux at the interface $x = x_{j+1/2}$ is determined by the intersection of the characteristic line $(x - x_{j+1/2}) - \lambda_k(\mu^n_{j+1/2})(t - t_{n+1}) = 0$ with the line $t = t_n$. When the Courant–Friedrichs–Lewy (C.F.L.) number:

$$\kappa_{j+1/2} = \mathrm{Max}\{\kappa_{j+1/2,k}, 1 \leqslant k \leqslant m\}, \tag{45}$$

where we set,

$$\begin{aligned}
&\kappa_{j+1/2,k} \equiv 0 \quad \text{for } \lambda_k(\mu^n_{j+1/2}) = 0,\\
&\kappa_{j+1/2,k} \equiv -\lambda_k(\mu^n_{j+1/2})\Delta x_{j+1}/(2\Delta t_n) \quad \text{for } \lambda_k(\mu^n_{j+1/2}) \leqslant 0,\\
&\kappa_{j+1/2,k} \equiv \lambda_k(\mu^n_{j+1/2})\Delta x_j/(2\Delta t_n) \quad \text{for } \lambda_k(\mu^n_{j+1/2}) \geqslant 0,
\end{aligned} \tag{46}$$

is larger than 1, as it is well known and understood, this strategy leads to a numerically unstable method. In such a case we develop another strategy which consists in determining the flux at the interface $x = x_{j+1/2}$ by making use of the intersection of the characteristic line $(x - x_{j+1/2}) - \lambda_k(\mu^n_{j+1/2})(t - t_{n+1}) = 0$ with the line $x = x_j$ when the characteristic C.F.L. number $\kappa_{j+1/2,k}$ is larger than $1/2$. This produce the following formula ([33]).

DEFINITION 3: *The numerical flux of the "implicit–explicit Characteristic flux" scheme is given by*

$$g^{CF}_j(v^n_j, v^n_{j+1}, v^{n+1}_j, v^{n+1}_{j+1}) = U_1 f(v^n_j) + U_2 f(v^n_{j+1}) + U_3 f(v^{n+1}_j) + U_4 f(v^{n+1}_{j+1}), \tag{47}$$

where the $m \times m$ matrices U_α, $\alpha \in \{1, 2, 3, 4\}$ are defined as follows:

$$U_\alpha \xi = \sum_{k=1}^{m} \omega^\alpha_k (\tilde{l}_k(\mu^n_{j+1/2}) \cdot \xi)\tilde{r}_k(\mu^n_{j+1/2}), \tag{48}$$

with $\mu^n_{j+1/2}$ given in Definition 1, $\tilde{l}_k$ and $\tilde{r}_k$ are the left and right eigenvectors of $\tilde{A}$ and the ω^α_k are defined according to the following conditions.

(i) *When* $0 < \lambda_k(\mu^n_{j+1/2}) \leqslant \Delta x_j/(2\Delta t_n)$, $\omega^1_k = 1, \omega^\alpha_k = 0$ *for* $\alpha \in \{2, 3, 4\}$.
(ii) *When* $-\Delta x_{j+1}/(2\Delta t_n) \leqslant \lambda_k(\mu^n_{j+1/2}) < 0$, $\omega^2_k = 1, \omega^\alpha_k = 0$ *for* $\alpha \in \{1, 3, 4\}$.
(iii) *When* $\lambda_k(\mu^n_{j+1/2}) = 0$, $\omega^1_k = \omega^2_k = 1/2, \omega^\alpha_k = 0$ *for* $\alpha \in \{3, 4\}$.
(iv) *When* $\lambda_k(\mu^n_{j+1/2}) > \Delta x_j/(2\Delta t_n)$, $\omega^1_k = \frac{\Delta x_j}{2\Delta t_n \lambda_k(\mu^n_{j+1/2})}, \omega^3_k = 1 - \omega^1_k, \omega^\alpha_k = 0$ *for* $\alpha \in \{2, 4\}$.
(v) *When* $\lambda_k(\mu^n_{j+1/2}) < -\Delta x_{j+1}/(2\Delta t_n)$, $\omega^2_k = \frac{\Delta x_{j+1}}{2\Delta t_n |\lambda_k(\mu^n_{j+1/2})|}, \omega^4_k = 1 - \omega^2_k, \omega^\alpha_k = 0$ *for* $\alpha \in \{1, 3\}$.

Now the implicit–explicit scheme for the discretization of (37) is:

$$v^{n+1}_j = v^n_j - \frac{\Delta t_n}{\Delta x_j}(\mathbf{I} + E(v^n_j))(g^{CF}_j(v^n_j, v^n_{j+1}, v^{n+1}_j, v^{n+1}_{j+1}) - g^{CF}_j(v^n_{j-1}, v^n_j, v^{n+1}_{j-1}, v^{n+1}_j)) + S(v^{n+1}_j). \tag{49}$$

Concerning the two schemes (44) and (49), we have the following stability result (we refer to [34] for the proof).

THEOREM 1: *Let a constant state* $\underline{v}$ *be given. We linearise equation* (17)*, with* $S \equiv 0$*, around this state which produces the linear equation*:

$$w_t + \underline{J} w_x + \underline{\tilde{C}} w_x + \underline{D} w_t = 0, \tag{50}$$

and assume that the matrix $(\mathbf{I} + \underline{D})^{-1}(\underline{J} + \underline{\tilde{C}})$ *is diagonalizable on* $\mathbb{R}$ (*hyperbolicity*).

- *The implicit–explicit characteristic flux scheme* (49)*, when applied to* (50)*, is unconditionally stable.*
- *Under the Courant–Friedrichs–Lewy condition* $|\underline{\Lambda}| \frac{\Delta t_n}{\Delta x_j} \leqslant 1$*, where* $\underline{\Lambda}$ *denotes the eigenvalue of* $(\mathbf{I} + \underline{D})^{-1}(\underline{J} + \underline{\tilde{C}})$ *with largest modulus, the explicit characteristic flux scheme* (44)*, when applied to* (50)*, is stable.*

2.3. Extension to the multidimensional case

Let us consider a system of m balance equations: $(v = (v_1, \ldots, v_m) \in \mathbb{R}^m)$

$$\frac{\partial v}{\partial t} + \nabla \cdot F(v) + \sum_{j=1}^{nd} \tilde{C}_j(v) \frac{\partial v}{\partial x_j} + D(v) \frac{\partial v}{\partial t} = \tilde{S}(v), \tag{51}$$

here $\nabla \cdot F(v) = \sum_{j=1}^{nd} \frac{\partial F^j(v)}{\partial x_j}$, where F^j maps G into $\mathbb{R}^m$, where G is an open subset of $\mathbb{R}^m$ corresponding to the physically admissible states. This equation is posed in a nd-dimensional domain Ω ($nd = 1, 2$ or 3 in practice). Here again we invert the matrix $\mathbf{I} + D(v)$ and obtain the following equation:

$$\frac{\partial v}{\partial t} + \nabla \cdot F(v) + \sum_{j=1}^{nd} C_j(v) \frac{\partial v}{\partial x_j} = S(v). \tag{52}$$

We assume that the computational domain Ω is decomposed in smaller volumes (the so-called control volumes) K: $\Omega = \bigcup_{K \in \mathcal{T}} K$ and consider first the case where $\Omega = \bigcup_{K \in \mathcal{T}} K$ is 'conformal', i.e. that it is a finite element triangulation of Ω. In practice one can use triangles for $nd = 2$ and tetrahedrons for $nd = 3$. The cell-centered finite volume approach for solving (52) consists in approximating the means

$$v_K(t) \equiv \frac{1}{\mathrm{vol}(K)} \int_K v(x, t)\, \mathrm{d}x, \tag{53}$$

where $\mathrm{vol}(K)$ denotes the nd-dimensional volume of K and $\mathrm{area}(A)$ stands for the $(nd-1)$-dimensional volume of an hypersurface A. Integrating (52) on K makes the normal fluxes, $F^v_{\partial K}$, appear

$$F^v_{\partial K}(t) = \int_{\partial K} F\big(v(\sigma, t)\big) \cdot \nu(\sigma)\, \mathrm{d}\sigma, \tag{54}$$

where ∂K is the boundary of K, $\nu(\sigma)$ the unit external normal on ∂K and $\mathrm{d}\sigma$ denotes the $(nd-1)$-volume element on this hypersurface.

The heart of the matter in finite volume methods consists in providing a formula for the normal fluxes $F^v_{\partial K}$ in terms of the $\{v_L\}_{L \in \mathcal{T}}$. Assuming that the control volumes K are polyhedra, as is most often the case, the boundary ∂K is the union of hypersurfaces $K \cap L$ where L belongs to the set $\mathcal{N}(K)$, the set of $L \in \mathcal{T}$, $L \neq K$, such that $K \cap L$ has positive $(nd-1)$-measure. We can therefore decompose the normal flux as a sum:

$F^v_{\partial K} = \sum_{L\in\mathcal{N}(K)} F_{K,L}$, where ($\nu_{K,L}$ points into L): $F_{K,L} = \int_{K\cap L} F(v(\sigma,t))\cdot\nu_{K,L}\,\mathrm{d}\sigma$. Inspired by the 1D-case, we take an approximation of $F_{K,L}$ in terms of v_K and v_L: $F_{K,L} \approx \mathrm{area}(K\cap L)\Phi(v_K, v_L; K, L)$, where Φ is the numerical flux that we construct by the following formula.

DEFINITION 4: *The numerical flux of the Finite Volume method with characteristic flux is obtained by the formula*

$$\Phi(v, w; K, L) = \frac{F(v)+F(w)}{2}\cdot\nu_{K,L} - U(v, w; K, L)\frac{F(w)-F(v)}{2}\cdot\nu_{K,L}, \tag{55}$$

when we take:

$$U(v, w; K, L) = \mathrm{sgn}\big(\tilde{A}_{\nu_{K,L}}\big(\mu(v, w; K, L)\big)\big), \tag{56}$$

where $\mu(v, w; K, L)$ is a mean between v_K and v_L which only depends on the geometry *of K and L*:

$$\mu(v, w; K, L) = \frac{\mathrm{vol}(K)v + \mathrm{vol}(L)w}{\mathrm{vol}(K)+\mathrm{vol}(L)}, \tag{57}$$

and

$$\tilde{A}_\nu(v) \equiv J_\nu(v)\big(\mathbf{I} + C_\nu(v)J_\nu(v)^{-1}\big), \tag{58}$$

where $J_\nu(v) = \frac{\partial F(v)\cdot\nu}{\partial v}$ and $C_\nu(v) = C(v)\cdot\nu \equiv \sum_{j=1}^{nd}\nu_j C_j(v)$.

Now that we have a formula for computing the numerical flux at an interface $K\cap L$, we have to discuss the discretization of the non-conservative products. Integrating equation (52) on a volume control K, leads to the question of the approximation of the integral $\int_K \sum_{j=1}^{nd} C_j(v)\frac{\partial v}{\partial x_j}\,\mathrm{d}x$. Following [35] (see also [36]), we take:

$$\int_K \sum_{j=1}^{nd} C_j(v)\frac{\partial v}{\partial x_j}\,\mathrm{d}x \approx \mathrm{area}(K\cap L)E_{K,L}\big(\Phi(v_K, v_L; K, L) - F(v_K)\cdot\nu_{K,L}\big), \tag{59}$$

where $E_{K,L}$ is the following $m\times m$ matrix:

$$E_{K,L} \equiv C_\nu(v_K)\tilde{A}_\nu(\mu)^{-1}, \quad \nu = \nu_{K,L} \text{ and } \mu = \mu_{K,L}. \tag{60}$$

This allows us to generalize the explicit scheme (44) to the multidimensional case as follows.

DEFINITION 5: *The explicit multidimensional characteristic flux for the approximation of equation* (52) *becomes*:

$$v_K^{n+1} = v_K^n - \frac{\Delta t_n}{\mathrm{vol}(K)}\sum_{L\in\mathcal{N}(K)} \mathrm{area}(K\cap L)\big(\mathbf{I} + E^n_{K,L}\big)\big(\Phi\big(v_K^n, v_L^n; K, L\big) - F\big(v_K^n\big)\cdot\nu_{K,L}\big) + S\big(v_j^n\big). \tag{61}$$

The implicit scheme (49) is generalized to the multidimensional case along the same lines and we refer to Boucker and Ghidaglia [35] for that purpose.

2.4. On the discretization of boundary conditions

So far we have not discussed the implementation of boundary conditions. This is a very important topic since they actually determine the solution. Let us consider the space discretization of the system (52) by our

cell centered finite volume method. For instance for the time explicit discretization we have the scheme (61). Of course this formula is not valid when K meets the boundary of Ω. When this occurs, we have to find the numerical flux $\Phi(v_K^n, K, \partial\Omega)$. In practice, this flux is not given by the physical boundary conditions and moreover, in general, (52) is an ill-posed problem if we try to impose either v or $F(v)\cdot\nu$ on $\partial\Omega$. This can be understood in a simple way by using the following linearization of this system:

$$\frac{\partial v}{\partial t} + \underline{A}_\nu \frac{\partial v}{\partial \nu} = 0, \tag{62}$$

where ν represents the direction of the external normal on $K \cap \partial\Omega$, $\underline{A}_\nu$ is the advection matrix:

$$\underline{A}_\nu \equiv \left.\frac{\partial F(v)\cdot\nu}{\partial v}\right|_{v=\underline{v}} + C_\nu(\underline{v}), \tag{63}$$

and $\underline{v}$ is the state around which the linearization is performed. When (52) is hyperbolic, the matrix $\underline{A}_\nu$ is diagonalizable on $\mathbb{R}$ and by a change of coordinates, this system becomes an uncoupled set of m advection equations:

$$\frac{\partial \xi_k}{\partial t} + c_k \frac{\partial \xi_k}{\partial \nu}, \quad k = 1, \ldots, m. \tag{64}$$

Here the c_k are the eigenvalues of $\underline{A}_\nu$ and according to the sign of these numbers, waves are going either into the domain Ω ($c_k < 0$) or out of the domain Ω ($c_k > 0$). Hence we expect that it is only possible to impose p conditions on $K \cap \partial\Omega$ where $p \equiv \sharp\{k \in \{1, \ldots, m\}$ such that $c_k < 0\}$.

Let us consider now a control volume K which meets the boundary $\partial\Omega$. We take $\underline{v} = v_K^n$ and write the previous linearization. We denote by x the coordinate along the outer normal so that (62) reads:

$$\frac{\partial v}{\partial t} + \underline{A}_\nu \frac{\partial v}{\partial x} = 0, \tag{65}$$

which happens to be the linearization of the 1D (i.e. when $nd = 1$) system. First we label the eigenvalues $c_k(\underline{v})$ of $\underline{A}_\nu$ by increasing order:

$$c_1(\underline{v}) \leqslant c_2(\underline{v}) \leqslant \cdots \leqslant c_p(\underline{v}) < 0 \leqslant c_{p+1}(\underline{v}) \cdots \leqslant c_m(\underline{v}). \tag{66}$$

(i) The case $p = 0$. In this case information comes from inside Ω and therefore we take:

$$\Phi\big(v_K^n, K, \partial\Omega\big) = F\big(v_K^n\big)\cdot\nu_K. \tag{67}$$

In the Computational Fluid Dynamics literature this is known as the 'supersonic outflow' case.

(ii) The case $p = m$. In this case information come from outside Ω and therefore we take:

$$\Phi\big(v_K^n, K, \partial\Omega\big) = \Phi_{\text{given}}, \tag{68}$$

where Φ_{given} are the given physical boundary conditions. In the Computational Fluid Dynamics literature this is known as the 'supersonic inflow' case.

(iii) The case $1 \leqslant p \leqslant m-1$. As already discussed, we need p scalar information coming from outside of Ω. Hence we assume that we have on physical ground p relations on the boundary:

$$g_l(v) = 0, \quad l = 1, \ldots, p. \tag{69}$$

Remark 7: The notation $g_l(v) = 0$ means that we have a relation between the components of v. However, in general, the function g_l is not given explicitly in terms of v. For example $g_l(v)$ could be the pressure which is not, in general, one of the components of v.

Since we have to determine the m components of $\Phi(v_K^n, K, \partial\Omega)$, we need $m - p$ supplementary scalar conditions. Let us write them as

$$h_l(v) = 0, \quad l = p+1, \ldots, m. \tag{70}$$

In general (69) are named as 'physical boundary conditions' while (70) are named as 'numerical boundary conditions'.

Then we take:

$$\Phi\big(v_K^n, K, \partial\Omega\big) = F(v) \cdot \nu_K, \tag{71}$$

where v is solution to (69)–(70) (see however Remark 10 and (77)).

Remark 8: The system (69)–(70) for the m unknowns $v \in G$ is a $m \times m$ nonlinear system of equations. We are going to study its solvability, see Theorem 2.

Let us first discuss the numerical boundary conditions (70). By analogy with what we did on an interface between two control volumes K and L, we take (recall that $\underline{v} = v_K^n$):

$$\tilde{l}_k(\underline{v}) \cdot \big(F(v) \cdot \nu_K\big) = \tilde{l}_k(\underline{v}) \cdot \big(F\big(v_K^n\big) \cdot \nu_K\big), \quad k = p+1, \ldots, m. \tag{72}$$

In other words, we set $h_k(v) \equiv \tilde{l}_k(v_K^n) \cdot (F(v) \cdot \nu_K) - \tilde{l}_k(v_K^n) \cdot (F(v_K^n) \cdot \nu_K)$. We have denoted by $(\tilde{l}_1(\underline{v}), \ldots, \tilde{l}_m(\underline{v}))$ a set of left eigenvectors of $\underline{\tilde{A}}_v$ (see (58)): ${}^t\underline{\tilde{A}}_v l_k(\underline{v}) = c_k l_k(\underline{v})$ and by $(r_1(\underline{v}), \ldots, r_m(\underline{v}))$ a set of right eigenvectors of $\underline{\tilde{A}}_v$: $\underline{\tilde{A}}_v r_k(\underline{v}) = c_k r_k(\underline{v})$. Moreover the following normalization is taken: $\tilde{l}_k(\underline{v}) \cdot \tilde{r}_p(\underline{v}) = \delta_{k,p}$.

According to Ghidaglia and Pascal [37] we have the following result on the solvability of (69)–(70).

THEOREM 2: *In the case* $1 \leqslant p \leqslant m-1$, *assume that* $c_{p+1}(\underline{v}) > 0$, *and*

$$\det_{1 \leqslant k,l \leqslant p} \left(\sum_{i=1}^{m} r_k^i(\underline{v}) \frac{\partial g_l}{\partial v_i}(\underline{v}) \right) \neq 0. \tag{73}$$

With the choice (72) *the nonlinear system* (69)–(70) *has one and only one solution* v, *for* $v - \underline{v}$ *and* $g_l(\underline{v})$ *sufficiently small.*

Remark 9: In this result we exclude the case where the boundary is characteristic, i.e. the case where one of the c_k is equal to 0. This case cannot be dealt at this level of generality. On the other hand, wall boundary conditions belong to this category. They can be discussed and handled directly on the physical system under consideration, see [37].

Remark 10: In practice, (69)–(70) are written in a parametric way. We have a set of m physical variables w (e.g. pressure, densities, velocities, ...) and we look for w satisfying:

$$g_l(w) = 0, \quad l = 1, \ldots, p, \tag{74}$$

$$\tilde{l}_k(\underline{v}) \cdot \Phi = \tilde{l}_k(\underline{v}) \cdot \big(F\big(v_K^n\big) \cdot \nu_K\big), \tag{75}$$

$$\Phi = F(w) \cdot \nu_K, \tag{76}$$

and then we take:

$$\Phi\big(v_K^n, K, \partial\Omega\big) = \Phi. \tag{77}$$

The system (74), (75) and (76) is then solved by Newton's method.

3. Applications to two fluid models

3.1. An isentropic model

In this section we particularize the previous analysis for the system of equations (1)–(4), in the 1D case, with the thermodynamics (11). The 3D case is similar and we refer to Boucker [36] for the details. We take $v = (\alpha_1\rho_1, \alpha_2\rho_2, \alpha_1\rho_1u_1, \alpha_1\rho_1u_2)$ and we set:

$$f(v) \equiv \big(\alpha_1\rho_1u_1, \alpha_2\rho_2u_2, \alpha_1\rho_1u_1^2 + \alpha_1(p-\pi), \alpha_2\rho_2u_2^2 + \alpha_2(p-\pi)\big), \tag{78}$$

$$C(v)v_x \equiv \big(0, 0, -(p-\pi)(\alpha_1)_x, -(p-\pi)(\alpha_2)_x\big), \tag{79}$$

$S(v) \equiv (0, 0, \alpha_1\rho_1g, \alpha_2\rho_2g)$. In (78) the function $\pi = \pi(t)$ has been artificially introduced and the reason for this is the subject of the following section.

3.1.1. On the introduction of the time-dependent parameter π

As discussed in Remark 4 there is some freedom in the choice between the conservative and non-conservative convective terms in the writing of (1) to (4) under the form (37). Equations (1)–(4) suggest taking: $v = (\alpha_1\rho_1, \alpha_2\rho_2, \alpha_1\rho_1u_1, \alpha_1\rho_1u_2)$, $f_0(v) \equiv (\alpha_1\rho_1u_1, \alpha_2\rho_2u_2, \alpha_1\rho_1u_1^2 + \alpha_1p, \alpha_2\rho_2u_2^2 + \alpha_2p)$ and $C_0(v)v_x \equiv (0, 0, -p(\alpha_1)_x, -p(\alpha_2)_x)$. Let us observe that the choice $f_1(v) \equiv (\alpha_1\rho_1u_1, \alpha_2\rho_2u_2, \alpha_1\rho_1u_1^2, \alpha_2\rho_2u_2^2)$ and $C_1(v)v_x \equiv (0, 0, \alpha_1p_x, \alpha_2p_x)$ is also possible. It leads to (12), (13) instead of (3), (4) which look simpler since they involve fewer terms.

Firstly, it is the first choice that should be preferred because when one fluid disappears, i.e. α_1 equals 0 or 1, the non-conservative term $C_0(v)v_x$ vanishes (this is physical information in the sense of Remark 4).

Secondly, formula (58) involves the matrix $E(v) = C(v)J(v)^{-1}$. A sufficient condition under which this matrix makes sense is that $J(v)$ is invertible, that is $d(v) \equiv \det J(v)$ never vanishes. It is this point (which is more of a mathematical nature) that leads us to the introduction of π in (78), (79). More precisely, with the notation of Remark 4, we take $g(v, t) \equiv (0, 0, -\alpha_1\pi, -\alpha_2\pi)$ where, for the time being, π is an arbitrary function of time t. In each context (see sections 3.1.4 and 3.2.3) we shall show how to construct this function in order that $E(v) = C(v)J(v)^{-1}$ makes sense.

Remark 11: Of course the total convection operator $A(v)$, see (14), is invariant under the introduction of π or more generally of $g(v, t)$ in (38). This means that although a term $\pi(\alpha_k)_x$ is added in the non-conservative terms, since a similar term is subtracted in the flux $f(v)$, in contrast with the terms (117)–(118), the introduction of π does not modify the characteristics (i.e. the eigenvalues of the convection operator) of the system under investigation.

3.1.2. Computations of the various matrices

Let us first compute the jacobian matrix $J(v) = \partial f(v)/\partial v$. We observe firstly that the two equations of state (11) allow us to consider that the α_i's and p are functions of v_1 and v_2 only:

$$\alpha_i = \alpha_i(v_1, v_2), \qquad p = p(v_1, v_2). \tag{80}$$

In the following, we shall denote by $q_{,j}$ the derivative of a quantity q with respect to v_j. Introducing the two speeds of sound c_j such that:

$$c_1^{-1} \equiv \sqrt{\frac{\mathrm{d}\rho_1}{\mathrm{d}p}}, \qquad c_2^{-1} \equiv \sqrt{\frac{\mathrm{d}\rho_2}{\mathrm{d}p}}, \tag{81}$$

we obtain the following expressions ($\alpha_{2,1} = -\alpha_{1,1}$, $\alpha_{1,2} = -\alpha_{2,2}$):

$$p_{,1} = \frac{\rho_2}{\alpha_1\rho_2 c_1^{-2} + \alpha_2\rho_1 c_2^{-2}}, \qquad p_{,2} = \frac{\rho_1}{\alpha_1\rho_2 c_1^{-2} + \alpha_2\rho_1 c_2^{-2}}, \tag{82}$$

$$\alpha_{1,1} = \frac{\alpha_2 c_2^{-2}}{\alpha_1\rho_2 c_1^{-2} + \alpha_2\rho_1 c_2^{-2}}, \qquad \alpha_{2,2} = \frac{\alpha_1 c_1^{-2}}{\alpha_1\rho_2 c_1^{-2} + \alpha_2\rho_1 c_2^{-2}}. \tag{83}$$

Remark 12: Equations (81) require that $\mathrm{d}\rho_i/\mathrm{d}p > 0$. This is a usual property of the equations of state (11).

Example 2: When fluid 1 is a perfect isentropic gas and fluid 2 is an incompressible fluid, the two equations of states (11) are:

$$p = A\rho_1^{\gamma}, \qquad \rho_2 = \text{ constant}. \tag{84}$$

In this case (80) becomes:

$$\alpha_1 = \frac{\rho_2 - v_2}{\rho_2}, \qquad \alpha_2 = \frac{v_2}{\rho_2}, \qquad p = A\left(\frac{v_1\rho_2}{\rho_2 - v_2}\right)^{\gamma}, \tag{85}$$

and we have $c_1^2 = \gamma p/\rho_1$, $c_2 = \infty$,

$$p_{,1} = \frac{\gamma p}{\alpha_1\rho_1}, \qquad p_{,2} = \frac{\gamma p}{\alpha_1\rho_2}, \qquad \alpha_{2,1} = \alpha_{1,1} = 0, \qquad \alpha_{1,2} = -\rho_2^{-1}, \qquad \alpha_{2,2} = \rho_2^{-1}. \tag{86}$$

We obtain for the Jacobian matrix $J(v)$ the following form:

$$J(v,\pi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_1 p_{,1} + \alpha_{1,1}(p-\pi) - u_1^2 & \alpha_1 p_{,2} + \alpha_{1,2}(p-\pi) & 2u_1 & 0 \\ \alpha_2 p_{,1} + \alpha_{2,1}(p-\pi) & \alpha_2 p_{,2} + \alpha_{2,2}(p-\pi) - u_2^2 & 0 & 2u_2 \end{pmatrix}. \tag{87}$$

On the other hand, according to (79), we have:

$$C(v,\pi) = -(p-\pi)C_0(v), \tag{88}$$

where:

$$C_0(v) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_{1,1} & \alpha_{1,2} & 0 & 0 \\ \alpha_{2,1} & \alpha_{2,2} & 0 & 0 \end{pmatrix}, \tag{89}$$

so that the matrix $A(v) = J(v,\pi) + C(v,\pi)$ is simply:

$$A(v) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_1 p_{,1} - u_1^2 & \alpha_1 p_{,2} & 2u_1 & 0 \\ \alpha_2 p_{,1} & \alpha_2 p_{,2} - u_2^2 & 0 & 2u_2 \end{pmatrix}. \tag{90}$$

Observe that the fact that the matrix $A(v)$ is independent of π results again from the fact that the decomposition chosen is consistent with the system (1) to (4) whatever the value of π is.

3.1.3. Domain of hyperbolicity

Let us go back to the general case where fluids 1 and 2 follow the general thermodynamics (11). The characteristic polynomial of the matrix $A(v)$ is:

$$P_4(v)(\lambda) = \lambda^4 - 2(u_1+u_2)\lambda^3 + \big(u_1^2 + 4u_1u_2 + u_2^2 - \alpha_1 p_{,1} - \alpha_2 p_{,2}\big)\lambda^2 + 2\big((\alpha_1 p_{,1} - u_1u_2)u_2 + (\alpha_2 p_{,2} - u_1u_2)u_1\big) + u_1^2u_2^2 - \alpha_1 p_{,1}u_1^2 - \alpha_2 p_{,2}u_2^2. \tag{91}$$

The following result is proved in Appendix A.

PROPOSITION 2: *The matrix $A(v)$ given in* (90) *is diagonalizable on $\mathbb{R}$ if and only if $(u_1-u_2)^2 > c_m^2$ where $c_m^2 \equiv ((\alpha_1 p_{,1})^{1/3} + (\alpha_2 p_{,2})^{1/3})^3$.*

Moreover if $0 < (u_1-u_2)^2 < c_m^2$, $A(v)$ is diagonalizable on $\mathbb{C}$ and has 2 real eigenvalues and two complex conjugate ones; while if $u_1 = u_2 \equiv u$, it has three real eigenvalues: $u - c_m$, u, $u + c_m$ but it is not diagonalizable since the eigenspace corresponding to u is one-dimensional (a 2×2 Jordan block appears in the reduced form of $A(v)$).

Remark 13: This result can be obtained without the use of Proposition 3 (Appendix A) either by studying the variations of $P_4(v)(\lambda)$ (Stewart and Wendroff [27]) or by computing explicitly the roots of $P_4(v)(\lambda)$ (Halaoua [38]). Here we prefer to rely on the results of the Appendix since they are very general and can be used whatever the degree of the characteristic polynomial is.

Remark 14: The hyperbolicity condition, $(u_1-u_2)^2 > c_m^2$, can also be expressed in terms of the two speeds of sound. We obtain easily that

$$c_m^2 = \frac{c_1^2 c_2^2}{\alpha_1\rho_2 c_2^2 + \alpha_2\rho_1 c_1^2}\big((\alpha_1\rho_1)^{1/3} + (\alpha_2\rho_2)^{1/3}\big)^3.$$

For example, if fluid 2 is incompressible ($c_2 = \infty$), we have

$$c_m^2 = c_1^2\left(\left(\frac{\rho_1}{\rho_2}\right)^{1/3} + \left(\frac{\alpha_2}{\alpha_1}\right)^{1/3}\right)^3.$$

In general, c_m is large compared to u_1 and u_2 and therefore the flow belongs to the non-hyperbolic region $0 \leqslant (u_1-u_2)^2 < c_m^2$.

3.1.4. On the determination of π

As discussed in section 3.1.1, the still undetermined function $\pi(t)$, has been introduced in order that:

$$d(v,\pi) \equiv \det\big(J(v,\pi)\big) \neq 0. \tag{92}$$

Remark 15: Condition (92) is sufficient for the matrix $E(v,\pi)$ to be defined. However a weaker condition suffices: let ${}^t\tilde{J}(v,\pi)$ denote the transpose of the comatrix of $J(v,\pi)$; it is sufficient that there exists a smooth $m\times m$ matrix $E(v,\pi)$ such that $C(v,\pi)^t\tilde{J}(v,\pi)=\det(J(v,\pi))E(v,\pi)$ (we use that $J(v,\pi)^t\tilde{J}(v,\pi)=\det(J(v,\pi))\mathbf{I}$). This remark will be used in the study of the 6 equation model, see (107)–(109).

By using (87), we obtain:

$$d(v,\pi)=a_4(v)-b_4(v)\pi, \tag{93}$$

with

$$a_4(v)\equiv u_1^2u_2^2-\alpha_2p_{,2}u_1^2-\alpha_1p_{,1}u_2^2+p\big(\alpha_{1,1}\big(p_{,2}-u_2^2\big)+\alpha_{2,2}\big(p_{,1}-u_1^2\big)\big), \tag{94}$$

$$b_4(v)\equiv\alpha_{2,2}p_{,1}+\alpha_{1,1}p_{,2}+\alpha_{1,2}u_1^2+\alpha_{2,1}u_2^2. \tag{95}$$

Introducing $a_4^0(v)\equiv p(\alpha_{1,1}p_{,2}+\alpha_{2,2}p_{,1})$, which is the value of $a_4(v)$ when $u_1=u_2=0$, we obtain, according to (82) and (83), that $a_4^0(v)=p(\alpha_1\rho_2c_1^{-2}+\alpha_2\rho_1c_2^{-2})^{-1}>0$ and $b_4(v)=(1-\alpha_1u_1^2c_1^{-2}-\alpha_2u_2^2c_2^{-2})(\alpha_1\rho_2c_1^{-2}+\alpha_2\rho_1c_2^{-2})^{-1}$ is in general positive. We choose $\pi(t)$ in order that $d(v,\pi)\geqslant a_4^0(v)$, i.e. we take $\pi(t)\leqslant\frac{a_4(v)-a_4^0(v)}{b_4(v)}$ that is:

$$\pi(t)=\inf_x\frac{a_4(v(x,t))-a_4^0(v(x,t))}{b_4(v(x,t))}. \tag{96}$$

Example 2 (continued): When fluid 1 is a perfect isentropic gas and fluid 2 is an incompressible fluid, see (84) we have:

$$a_4(v)=\frac{\gamma p^2+u_1^2u_2^2\rho_1\rho_2-p(\gamma\alpha_1\rho_2u_2^2+(\gamma\alpha_2+\alpha_1)\rho_1u_1^2)}{\rho_1\rho_2\alpha_1}, \tag{97}$$

$$b_4(v)=\frac{\gamma p-\alpha_1\rho_1u_1^2}{\rho_1\rho_2\alpha_1},\qquad a_4^0(v)=\frac{\gamma p^2}{\rho_1\rho_2\alpha_1}, \tag{98}$$

and therefore:

$$\pi(t)=\inf_x\frac{u_1^2u_2^2\rho_1\rho_2-p(\gamma\alpha_1\rho_2u_2^2+(\gamma\alpha_2+\alpha_1)\rho_1u_1^2)}{\gamma p-\alpha_1\rho_1u_1^2}(x,t). \tag{99}$$

3.2. A 6 equation model

In this section we particularize the analysis of section 2 for the system of equations (1)–(6), in the 1D case, with the thermodynamics (7). This system can be written as (51) with the notations introduced in (15) and (18)–(20) but, for similar reasons as in the previous section, we introduce again a parameter $\pi(t)$ and supplement (78) with $f_5=\alpha_1\rho_1H_1u_1$, $f_6=\alpha_2\rho_2H_2u_2$ and take:

$$\tilde{C}(v)v_x\equiv\big(0,0,-(p-\pi)(\alpha_1)_x,-(p-\pi)(\alpha_2)_x,0,0\big), \tag{100}$$

instead of (19).

We shall mainly consider here the case of a simplified thermodynamics: the case where fluid 2 is incompressible, i.e. ρ_2 is constant.

3.2.1. Computation of the various matrices

The major simplification in this case is that the α_j are easily expressed in terms of v:

$$\alpha_1 = \frac{\rho_2 - v_2}{\rho_2}, \qquad \alpha_2 = \frac{v_2}{\rho_2}. \tag{101}$$

It follows that we can write the two terms $p\frac{\partial \alpha_j}{\partial t}$ which appear in (5) and (6) as $p\frac{\partial \alpha_1}{\partial t} = \frac{p}{\rho_2}\frac{\partial v_4}{\partial x}$, $p\frac{\partial \alpha_2}{\partial t} = -\frac{p}{\rho_2}\frac{\partial v_4}{\partial x}$ (we have used (2)). This calculation corresponds to the inversion of the matrix $\mathbf{I} + D(v)$ and we arrive now at the form (37) where $S(v)$ is given in (16) and:

$$C(v,\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{p-\pi}{\rho_2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{p-\pi}{\rho_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p}{\rho_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{p}{\rho_2} & 0 & 0 \end{pmatrix}. \tag{102}$$

The Jacobian matrix $J(v) = \partial f(v)/\partial v$ has the following form:

$$J(v,\pi) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ J_{31} & J_{32} & J_{33} & 0 & J_{35} & 0 \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} & 0 \\ K_{51}u_1 & K_{52}u_1 & J_{53} & 0 & K_{55}u_1 & 0 \\ K_{61}u_2 & K_{62}u_2 & J_{63} & J_{64} & K_{65}u_2 & u_2 \end{pmatrix}, \tag{103}$$

where:

$$J_{31} = \alpha_1 p_{,1} - u_1^2, \qquad J_{32} = \alpha_1 p_{,2} - \frac{p-\pi}{\rho_2}, \qquad J_{33} = \alpha_1 p_{,3} + 2u_1, \qquad J_{35} = \alpha_1 p_{,5},$$

$$J_{41} = \alpha_2 p_{,1}, \qquad J_{42} = \alpha_2 p_{,2} + \frac{p-\pi}{\rho_2} - u_2^2, \qquad J_{43} = \alpha_2 p_{,3}, \qquad J_{44} = 2u_2, \qquad J_{45} = \alpha_2 p_{,5},$$

$$K_{51} = \alpha_1 p_{,1} - H_1, \qquad K_{52} = \alpha_1 p_{,2} - \frac{p}{\rho_2}, \qquad J_{53} = \alpha_1 p_{,3} u_1 + H_1, \qquad K_{55} = 1 + \alpha_1 p_{,5},$$

$$K_{61} = \alpha_2 p_{,1}, \qquad K_{62} = \alpha_2 p_{,2} + \frac{p}{\rho_2} - H_2, \qquad J_{63} = \alpha_2 p_{,3} u_2, \qquad J_{64} = H_2, \qquad K_{65} = \alpha_2 p_{,5}.$$

Let us write the equation of state of fluid 1 as $p = p(\rho_1, e_1)$. Then

$$p(v) = p\left(\frac{\rho_2 v_1}{\rho_2 - v_2}, \frac{v_5}{v_1} - \frac{v_3^2}{2v_1^2}\right), \tag{104}$$

and the coefficients of J are:

$$J_{31} = \frac{(\rho_1 p_{,\rho} - E_1 p_{,e})}{\rho_1} - u_1^2, \qquad J_{32} = \frac{p-\pi}{\rho_2} + \frac{\rho_1 p_{,\rho}}{\rho_2}, \qquad J_{33} = u_1\left(2 - \frac{p_{,e}}{\alpha_1}\right),$$

$$J_{35} = \frac{p_{,e}}{\rho_1}, \qquad J_{41} = \alpha_2\left(\frac{p_{,\rho}}{\alpha_1} - \frac{E_1 p_{,e}}{\alpha_1 \rho_1}\right), \qquad J_{42} = \frac{p-\pi}{\rho_2} + \frac{\alpha_2 \rho_1 p_{,\rho}}{\alpha_1 \rho_2} - u_2^2,$$

$$J_{43} = -\frac{\alpha_2 u_1 p_{,e}}{\alpha_1 \rho_1}, \qquad J_{44} = 2u_2, \qquad J_{45} = \frac{\alpha_2 p_{,e}}{\alpha_1 \rho_1}, \qquad K_{51} = p_{,\rho} - H_1 - \frac{(E_1 - u_1^2) p_{,e}}{\rho_1}, \tag{105}$$

$$K_{52} = \frac{\rho_1 p_{,\rho} - p}{\rho_2}, \qquad J_{53} = H_1 - \frac{p_{,e} u_1^2}{\rho_1}, \qquad K_{55} = 1 + \frac{p_{,e}}{\rho_1}, \qquad K_{61} = \frac{\alpha_2}{\alpha_1 \rho_1}(\rho_1 p_{,\rho} - E_1 p_{,e}),$$

$$K_{62} = \frac{\alpha_2 \rho_1 p_{,\rho}}{\alpha_1 \rho_2} - E_2, \qquad J_{63} = \alpha_2 \frac{p_{,e}}{\alpha_1 \rho_1} u_1 u_2, \qquad J_{64} = H_2, \qquad K_{65} = \alpha_2 \frac{p_{,e}}{\alpha_1 \rho_1}.$$

3.2.2. Domain of hyperbolicity

The characteristic polynomial of the matrix $A(v)$ is here:

$$P_6(v)(\lambda) = P_4(v)(\lambda)(\lambda - u_1)(\lambda - u_2). \tag{106}$$

The polynomial $P_4(v)$ is given in (91), where the computations have to be made with an equation of state for phase 1 with a frozen entropy s_1. Hence according to Proposition 2 and Remark 14, the hyperbolicity condition will simply become

$$(u_1 - u_2)^2 > c_1^2 \left(\left(\frac{\rho_1}{\rho_2} \right)^{1/3} + \left(\frac{\alpha_2}{\alpha_1} \right)^{1/3} \right)^3, \quad c_1^2 \equiv \left(\frac{\partial p}{\partial \rho_1} \right)_{s_1}.$$

3.2.3. On the determination of π

In order to choose $\pi(t)$, we compute again the determinant of $J(v, \pi)$. We find here:

$$d(v, \pi) = \det\big(J(v, \pi)\big) = u_1 u_2 \big(a_6(v) - b_6(v)\pi\big), \tag{107}$$

where $a_6(v)$ and $b_6(v)$ are smooth functions of the v_i's, see (110). On the other hand, we can also find a smooth matrix $E_0(v, \pi)$ such that:

$$C(v, \pi)^t \tilde{J}(v, \pi) = u_1 u_2 E_0(v, \pi), \tag{108}$$

and therefore, provided that we can ensure that π is such that $(a_6(v) - b_6(v)\pi)$ cannot vanish, we can define $E(v, \pi)$ by

$$E(v, \pi) \equiv \big(a_6(v) - b_6(v)\pi\big)^{-1} E_0(v, \pi). \tag{109}$$

Making use of (105), we find for $a_6(v)$ and $b_6(v)$:

$$\begin{aligned} a_6(v) &= \alpha_1 \rho_1^2 \rho_2 u_1^2 u_2^2 + \big(p - \alpha_1 \rho_2 u_2^2\big) p p_{,e} + \rho_1^2 \big(p\big(p_{,\rho} - \alpha_1 u_1^2\big) - p_{,\rho}\big(\alpha_1 \rho_2 u_2^2 + \alpha_2 \rho_1 u_1^2\big)\big), \\ b_6(v) &= p p_{,e} + \alpha_2 \rho_1 u_1^2 p_{,e} + \rho_1^2 p_{,\rho} - \alpha_1 \rho_1^2 u_1^2. \end{aligned} \tag{110}$$

Example 3: When fluid 1 is a polytropic gas: $p(\rho, e) \equiv (\gamma - 1)\rho e$ where $\gamma > 1$, we have $p_{,e} = (\gamma - 1)\rho$ and $p_{,\rho} = (\gamma - 1)e$ hence:

$$\begin{aligned} a_6(v) &= \gamma \rho_1 p^2 + \alpha_1 \rho_1^2 \rho_2 u_1^2 u_2^2 - \rho_1 p\big(\gamma \alpha_1 \rho_2 u_2^2 + \rho_1 u_1^2\big), \\ b_6(v) &= \rho_1 \big(2(\gamma - 1)p + (\gamma \alpha_2 - 1)\rho_1 u_1^2\big). \end{aligned} \tag{111}$$

Let $a_6^0(v)$ be the value of $a_6(v)$ when $u_1 = u_2 = 0$, i.e. $a_6^0(v) = p(p p_{,e} + \rho_1^2 p_{,\rho})$. This number is in general positive (see, e.g., (111)). So, in order to impose that $(a_6(v) - b_6(v)\pi)$ does not vanish, we choose $\pi(t)$ such

that $(a_6(v) - b_6(v)\pi) \geqslant a_6^0(v)$ that is (observe that, in general, $b_6(v) > 0$):

$$\pi(t) = \inf_x \frac{a_6(v(x,t)) - a_6^0(v(x,t))}{b_6(v(x,t))}. \tag{112}$$

It remains for us to construct the matrix E_0 which appears in (108). Since the coefficients of the matrix ${}^t\tilde{J}(v,\pi)$ are polynomial in terms of those of $J(v,\pi)$, the computation of $C(v,\pi)^t\tilde{J}(v,\pi)$ is straightforward using a computer algebra program and one verifies that indeed all the terms of this matrix factorizes u_1u_2. Hence the formula (109), which is indeed well defined thanks to our choice of π.

Remark 16: Since $\nabla_x\pi(t) = 0$, at the continuous level the introduction of π does not modify the equations which are solved. However, at the discrete level, there is a difference which has to do with the discretization of the non-conservative product $p\nabla\alpha_1$. We have checked the influence of this choice by comparing numerically the effect of taking π, 10π, $10^2\pi$ and $10^3\pi$ (in this case π was negative). This has produced absolutely no visible effect on the solution.

3.2.4. More on the hyperbolicity of the convection operator

Many of the authors who have considered the model (1)–(6) have proposed a modification of the convection operator in order to produce a matrix $A(v)$ (see (14)) which has real eigenvalues, at least in the numerical range where the computations are performed.

In general this goal is achieved by adding forces in the balance of momentum which could physically correspond to transfers of momentum between the two phases. Let us denote by $f_{2\to1}$ the force that will appear in the equation (3) then $f_{1\to2} = -f_{2\to1}$ and the work of these forces will appear in the corresponding equations of energy:

$$\frac{\partial(\alpha_1\rho_1u_1)}{\partial t} + \operatorname{div}\big(\alpha_1(\rho_1u_1\otimes u_1 + p\mathbf{I})\big) - p\nabla\alpha_1 = \alpha_1\rho_1g + f_{2\to1}, \tag{113}$$

$$\frac{\partial(\alpha_2\rho_2u_2)}{\partial t} + \operatorname{div}\big(\alpha_2(\rho_2u_2\otimes u_2 + p\mathbf{I})\big) - p\nabla\alpha_2 = \alpha_2\rho_2g + f_{1\to2}, \tag{114}$$

$$\frac{\partial(\alpha_1\rho_1E_1)}{\partial t} + \operatorname{div}(\alpha_1\rho_1H_1u_1) + p\frac{\partial\alpha_1}{\partial t} = (\alpha_1\rho_1g + f_{2\to1})\cdot u_1, \tag{115}$$

$$\frac{\partial(\alpha_2\rho_2E_2)}{\partial t} + \operatorname{div}(\alpha_2\rho_2H_2u_2) + p\frac{\partial\alpha_2}{\partial t} = (\alpha_2\rho_2g + f_{1\to2})\cdot u_2. \tag{116}$$

From the physical point of view, these forces correspond, e.g., to added mass forces or interfacial pressure. In the later case, we have:

$$f_{2\to1} = -(p - p_{\text{interface}})\nabla\alpha_1, \tag{117}$$

and, for example, Bestion [39] has proposed to take:

$$(p - p_{\text{interface}}) \equiv \delta\frac{\alpha_1\alpha_2\rho_1\rho_2}{\alpha_1\rho_2 + \alpha_2\rho_1}(u_1 - u_2)^2. \tag{118}$$

Actually, δ is equal to 1 in [39], but here we introduce this parameter in order to study its influence on the solution (see *figure 4*).

3.2.5. The general thermodynamics

In the previous sections, we dealt with the case where one of the two fluids is incompressible (let say fluid 2). This case is simpler as far as analytical computations are concerned (see, e.g., (101)). On the other hand this case is singular in the limit $\alpha_1 = 0$ since, as it is well known, the motion of a single incompressible fluid leads to the non-evolutionary equation $\operatorname{div} u = 0$. Hence if we want to consider the whole range $0 < \alpha_1 < 1$, we have to deal with two compressible fluids. This actually makes the analytical computations more complicated but still the system obtained falls into the framework of equation (37).

4. A numerical illustration

The goal of this section is not to present a wide variety of numerical results in order to convince the reader of the wide applicability of the method we have introduced in this paper. As already noticed, this method was presented in 1995 ([24]) in a 6 page note. We have waited on purpose for sometime before publishing an extended paper describing in details this finite volume method in order to validate it on large variety of cases including 3D computations. We refer to the report [25] concerning both numerical results and references including numerical results.

Here we will just deal with what is known as Ransom faucet flow ([40]) which is a numerical benchmark. This test case is well referenced and analytically simple. It has also the great interest that it contains some of the important features in the field at least from the numerical point of view. Moreover analytical solution is available for comparison, [40]. The solution mainly consists of a shock wave that travels under the effect of gravity.

The continuous model is (1) to (6), in dimension 1, and we consider the case where phase 1 is a perfect polytropic gas (Example 3) with $\gamma = 1.4$ and phase 2 is incompressible: ρ_2 is constant, see section 3.2. All the numerical values given below are expressed in the International System of Units. The equations are posed for x between 0 and 12 and the gravity is taken as $g = 10$. The boundary conditions are as follows:

$$\alpha_1(0,t) = 0.2, \qquad u_1(0,t) = 0, \qquad u_2(0,t) = 10, \tag{119}$$

$$h_2^s(0,t) = 209280, \tag{120}$$

$$p(12,t) = 10^5, \quad \text{when } u_1(12,t) \geqslant 0, \tag{121}$$

$$p(12,t) = 10^5, \qquad h_1^s(12,t) = 324594, \quad \text{when } u_1(12,t) < 0, \tag{122}$$

h_2^s in (120) corresponds to water at a temperature of 323.15 K and atmospheric pressure, while h_1^s in (122) corresponds to air in the same conditions. Note that the h_i^s's denote the specific enthalpies of each fluid and should not be confused with the boundary conditions (70). Actually, they belong to the set of physical boundary conditions (69).

In *figure 1* we show the time evolution of the 'void fraction' α_1 in the case where 192 equally spaced cells are used ($\Delta x = 12/192 = 0.0625$). In *figure 2*, we show that the computed front is sharper and sharper as the number of cells is increased. However, since no physical dissipation is included in this model, an instability appears for Δx too small. This is due to the non-hyperbolic character of the convection operator. If we want to use very fine meshes without including physical dissipation, we have to turn to an hyperbolic convection operator. For example using the correction (117)–(118), with $\delta = 1.01$, allows us to use more than a thousand of cells, see *figure 3* that displays the exact solution (shock front) and the computed solutions with 768 and 1536 cells. The higher the number of cells, sharper is the calculated front.

Our next goal was to study the effect of this correction on the solution. In order to do so, one needs a method that allows to consider both hyperbolic and non-hyperbolic convection operators. This one of the features of

the method proposed in this paper. The results are shown in *figure 4*. It appears that the correction term, besides the fact that it renders the convection operator hyperbolic, induces a numerical dissipation. This dissipation is quite low when $\delta = 1.01$. However there is no guarantee that for any flow, this value will lead to a hyperbolic convection operator.

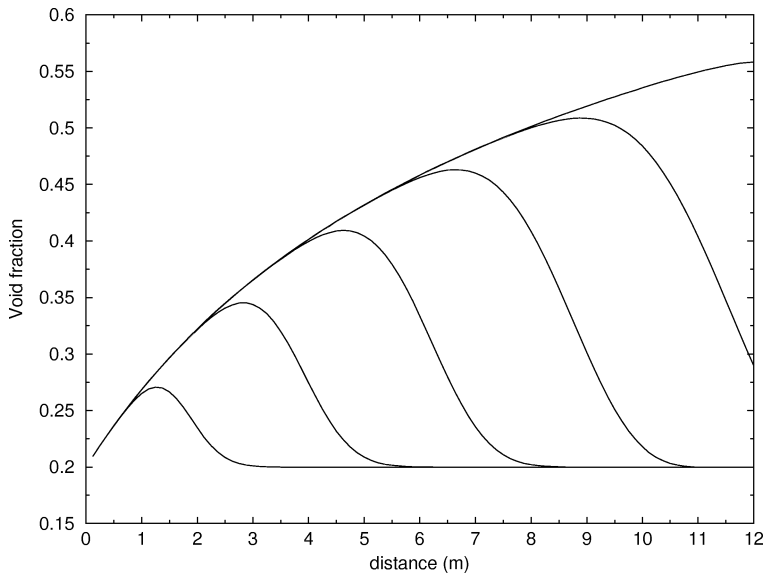

Figure 1. Propagation of the solution according to time (96 cells).

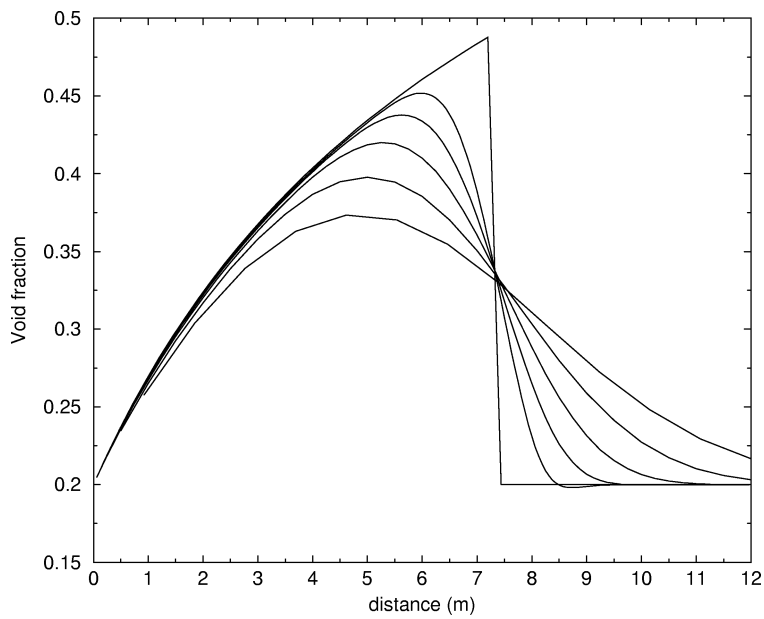

Figure 2. Influence of the discretization (from 12 to 192 cells).

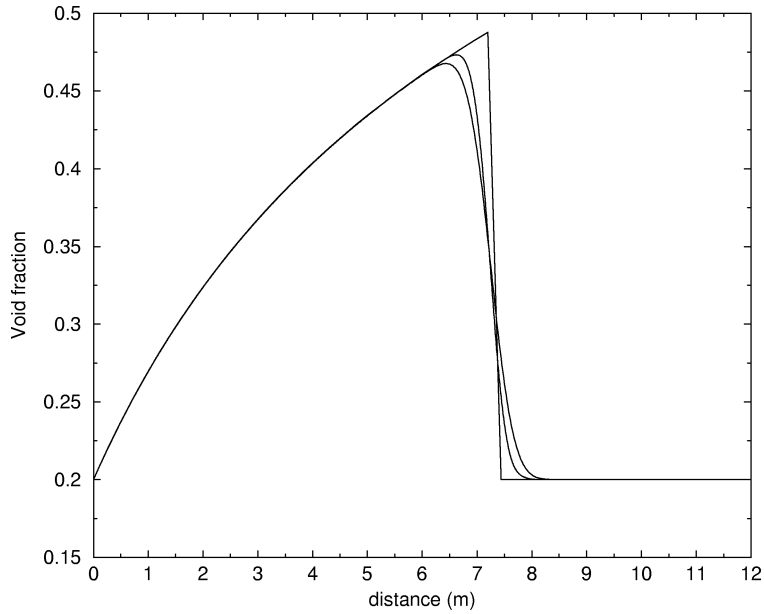

Figure 3. Exact solution, and computed ones with 768 and 1536 cells.

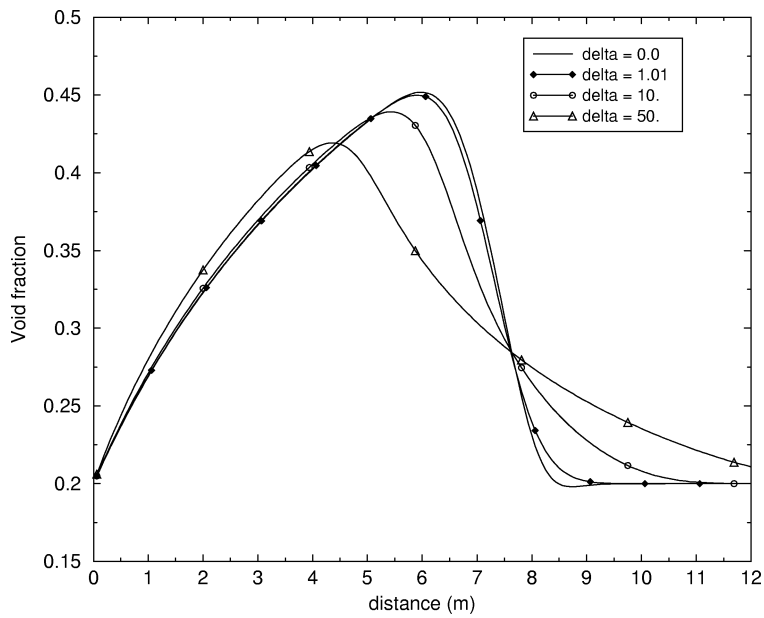

Figure 4. Influence of the hyperbolicity parameter (192 cells).

5. Further developments

The model (1)–(6) is known as the basic two fluid model in the sense that it does not contain all the terms related to the transfers between the phases and those which are related to dissipation phenomena. These two categories of terms are of a different nature. The later involve second-order gradients and can be

discretized mainly following to two different ways. The first one makes use of gradient reconstruction (see, e.g., Kröner [41]): one constructs by an algebraic formula a gradient of the numerical solution from its finite volume representation. The second way consists in translating the finite volume degrees of freedom into finite element ones and then apply a variational approach for the discretization of diffusion terms. Here again there are two strategies. The first one keeps the finite volume degrees of freedom unchanged and introduces a dual mesh on which these degrees of freedom are those of the finite element discretization (see, e.g., Guillard [42] and the references therein). This strategy is not practically adapted to the case of 3D unstructured meshes due to the dual mesh whose construction might be very difficult if impossible. The second strategy, proposed by Pascal and Ghidaglia [43] and [44] consists in keeping the same mesh but changing the degrees of freedom thanks to 'footbridges'. This method gives good results in the present context (see Boucker [36]).

Concerning the terms related with transfers between the phases (mass, momentum or mass transfers) they can be classified into two categories. The first one includes those that do not involve derivatives, such as, for example, the drag force which is proportional to $|u_1 - u_2|(u_1 - u_2)$ or some stiff terms that correspond to changes of phase. These are called sources terms and are included in the abstract equation (51) in the term $\tilde{S}(v)$. In general these terms are discretized implicitly in time and one also uses an upwind discretization in space (see Alouges et al. [31]). Finally the terms that contain first-order derivatives are in general incorporated into the non-conservative products $\sum_{j=1}^{nd} \tilde{C}_j(v)\frac{\partial v}{\partial x_j} + D(v)\frac{\partial v}{\partial t}$ in equation (51).

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Appendix A: On the hyperbolicity of the convection operator

Let $P = X^n + \sum_{j=0}^{n-1} a_j X^j$ be a polynomial (with real coefficients) of degree n. It's Sturm sequence is defined as follows:

$$P_0 = P, \qquad P_1 = P', \qquad P_m = -\text{remainder}(P_{m-2}, P_{m-1}), \tag{123}$$

where remainder(P, Q) denotes the remainder in the Euclidean division of P by Q. The polynomial P_k is of degree $n - k$, and we denote by ξ_k the coefficient of X^{n-k} in P_k. Using Sturm's theorem (see, e.g., Corollary 3.8, page 10 in Roy [45]) we have the following result.

PROPOSITION 3: *With the previous notations, the polynomial P has n real roots if and only if all the ξ_k are positive.*

Remark 17: (i) Since the computation of the Sturm sequence is straightforward (and uses only additions, multiplications and divisions), this result is applicable whatever the degree of the polynomial is. Note however that the bitlength of the coefficients in the sequence can increase dramatically (see, e.g., Example 3.10, page 11 in Roy [45]).

(ii) These computations can be done on a polynomial whose coefficients depend on parameters, using if necessary a computer algebra program.

Let us now apply this proposition to the characteristic polynomial which appears in (91). In order to make the computations more simple, we set $\lambda \equiv X + \frac{u_1+u_2}{2}$ and $u_r \equiv u_1 - u_2$. Hence we obtain:

$$P_0 = P = X^4 + qX^2 + rX + s, \tag{124}$$

where $q \equiv -(p_2\alpha_2 + p_1\alpha_1 + 1/2u_r^2)$, $r \equiv (p_2\alpha_2 - p_1\alpha_1)u_r$ and $s \equiv (u_r^2 - 4(p_1\alpha_1 + p_2\alpha_2))u_r^2/16$. It follows that:

$$P_1 = 4X^3 + 2qX + r, \tag{125}$$

$$P_2 = -1/2qX^2 - 3/4rX - s, \tag{126}$$

$$\tilde{P}_3 = \big(8qs - 9r^2 - 2q^3\big)X - q^2r - 12rs, \tag{127}$$

$$\tilde{P}_4 = q^2\big(256s^3 - 128q^2s^2 + 144qsr^2 + 16q^4s - 27r^4 - 4r^2q^3\big), \tag{128}$$

where $\tilde{P}_3$ (respectively $\tilde{P}_4$) is obtained by multiplying P_3 (resp. P_4) by a positive coefficient. Hence, following Proposition 3, the polynomial P will have exactly 4 real roots if and only if:

$$q < 0, \qquad 8qs - 9r^2 - 2q^3 > 0 \quad \text{and} \quad \tilde{P}_4 > 0. \tag{129}$$

In view of (82), we have $q < 0$. After some straightforward but somewhat cumbersome computations, one finds that the two other conditions will be fulfilled if and only if $u_r^2 > ((\alpha_1 p_{,1})^{1/3} + (\alpha_2 p_{,2})^{1/3})^3$. This leads to the first part of Proposition 2. The study of the case where (129) is not satisfied follows the same lines: in this case ([45]) what matters is the number of change of sign in the sequence $(\xi_1, \xi_2, \xi_3, \xi_4)$. Details are left to the reader.

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